Properties of the class of power-logistic maps

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In a study of the class of power-logistic maps, each of which consists of a power-law branch $1-r|x_n|^z$ for negative values of x_n and a quadratic (logistic) branch $1-rx_n^2$ for positive values of x_n with parameter $r\in(0,2]$ and exponent $z\in(0,2]$, we found the following: (i) In the chaotic region, there are stable cycles whose periods can be regarded to form an arithmetic progression known as pattern A (PA). (ii) As z decreases, PA is more prominent; nevertheless, it still exists in the logistic map. (iii) The first term of PA is a function of z: as z decreases, it either stays constant or increases by 2. (iv) As z decreases, a given PA term begins to appear at a smaller r value. (v) When z is sufficiently large, the range of a PA term increases as z decreases. (vi) Between two consecutive PA terms, there are structures such as period-doubled cycles of the PA terms, other stable cycles, and a chaotic subregion. (vii) As z decreases, the chaotic subregion between any two consecutive PA terms shrinks, which may result in a loss of fine structures. [S1063-651X(96)12511-3]

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I. INTRODUCTION

Recently, there has been much interest in the study of discontinuous maps, consisting of piecewise continuous sections, as they have many interesting properties such as the existence of a renormalization group with periodic behavior [1], the existence of fine structures, precision-dependent periods, and summation rules governing the inverse cascades [2–5]; they can also be used for modeling some physical systems [6,7].

Do other asymmetric unimodal maps with two different branches also have interesting fine structures, and if they do, how dependent are their properties on the details of the maps? To answer these questions, we shall study the following class of one-dimensional unimodal exponent-asymmetric maps:

$$g(x_n)=\begin{cases}1-r|x_n|^z & \text{if } x_n<0\\ 1-rx_n^2 & \text{if } x_n\geqslant 0,\end{cases} \tag{1}$$

which has a ''logistic'' branch and a ''power-law'' branch, with exponent $z\in(0,2]$ and parameter $r\in(0,2]$, for the presence of fine structures and their dependence on z.

This map, to be known as the power-logistic map, reduces to the logistic map when $z=2$ and to the linear-logistic map when $z=1$ [8]. Further, when $z=2$, the power-logistic map is a C^∞ function, while for $1<z<2$, it is a C^1 function, but for $z\leqslant 1$, the slope is not continuous at $x=0$. Thus this map is interesting in its own right and it can be used to study the effect of introducing a discontinuity in the slope at the maximum. Moreover, the power-logistic map has some physical relevance since a special case of it, namely, the linear-logistic map, can be used for modeling some dynamical systems such as the impact oscillator [9] and the electronic forced oscillator [10].

In this paper, we shall report our numerical study of the behavior of the map $g(x)$ with selected values of z in the range $(0,2]$ and with r over its complete range. Thus for each selected value of z, we can determine the range of r over which the orbits have period n, $\Delta r(n)$, which is equal to $r(n,\max)-r(n,\min)$ where $r(n,\max)$ and $r(n,\min)$ denote, respectively, the maximum and minimum values of r at which the map has an n cycle.

An outline of this paper is as follows: The existence and properties of pattern A, which is an arithmetic progression whose terms are the periods of certain stable cycles in the chaotic region of the map with a given value of z, will be presented in Sec. II. The variation of some other properties of this map with z will be given in Sec. III. The next section will include a discussion on the existence of pattern A (PA) in the logistic map. Finally, a brief summary and discussion will be presented in Sec. V.

II. EXISTENCE AND PROPERTIES OF PATTERN *A*

A. Existence of pattern *A*

For the power-logistic map with a given value of z, the nature of the orbits as a function of r can be determined numerically. This information is summarized in the bifurcation diagram, a typical one being shown in Fig. 1 for $z=0.8$. From such a bifurcation diagram and the graph of the Lyapunov exponent λ as a function of the parameter r for a given value of z, we are able to distinguish some stable cycles in the chaotic region. These have special features in the λ-r graph, namely, a smoothly ascending structure that rises from a negative value of λ to a peak value of 0. It is found that the periods of these stable cycles form an arithmetic progression. For example, in the λ-r graph with $z=0.4$ shown in Fig. 2(a), the third and subsequent labeled cycles, which lie in the chaotic region, have the respective periods 8,10,12,14, . . . , which form an arithmetic progression; each term is associated with a smoothly ascending structure that rises from a negative value of λ to a peak value of 0. We shall refer to the members of this arithmetic progression as ''terms.''

The first two labeled periodic orbits in the above figure, namely, the 2 cycle and the 6 cycle are located within the

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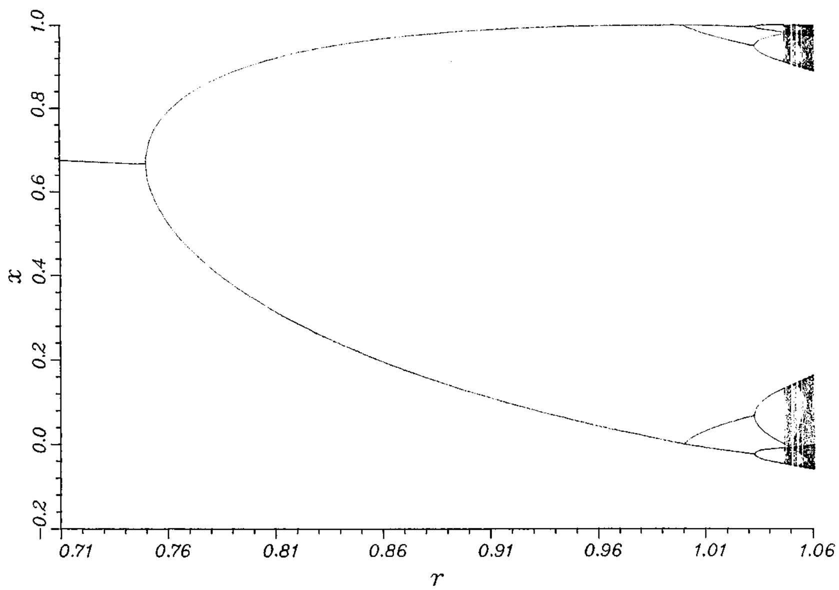

FIG. 1. Bifurcation diagram of the power-logistic map with $z=0.8$.

periodic region; the chaotic region that lies just before the 8 cycle, is barely visible here. Though the 6 cycle has the same rising structure, it is not regarded as part of the same series since this cycle lies in the periodic region. Further, the cycle associated with the descending region to the right of each peak at $\lambda=0$ is the period-doubled cycle of the periodic orbit associated with the ascending structure to the left of that peak.

As other examples of the special structure associated with these terms, and their membership of an arithmetic progres-

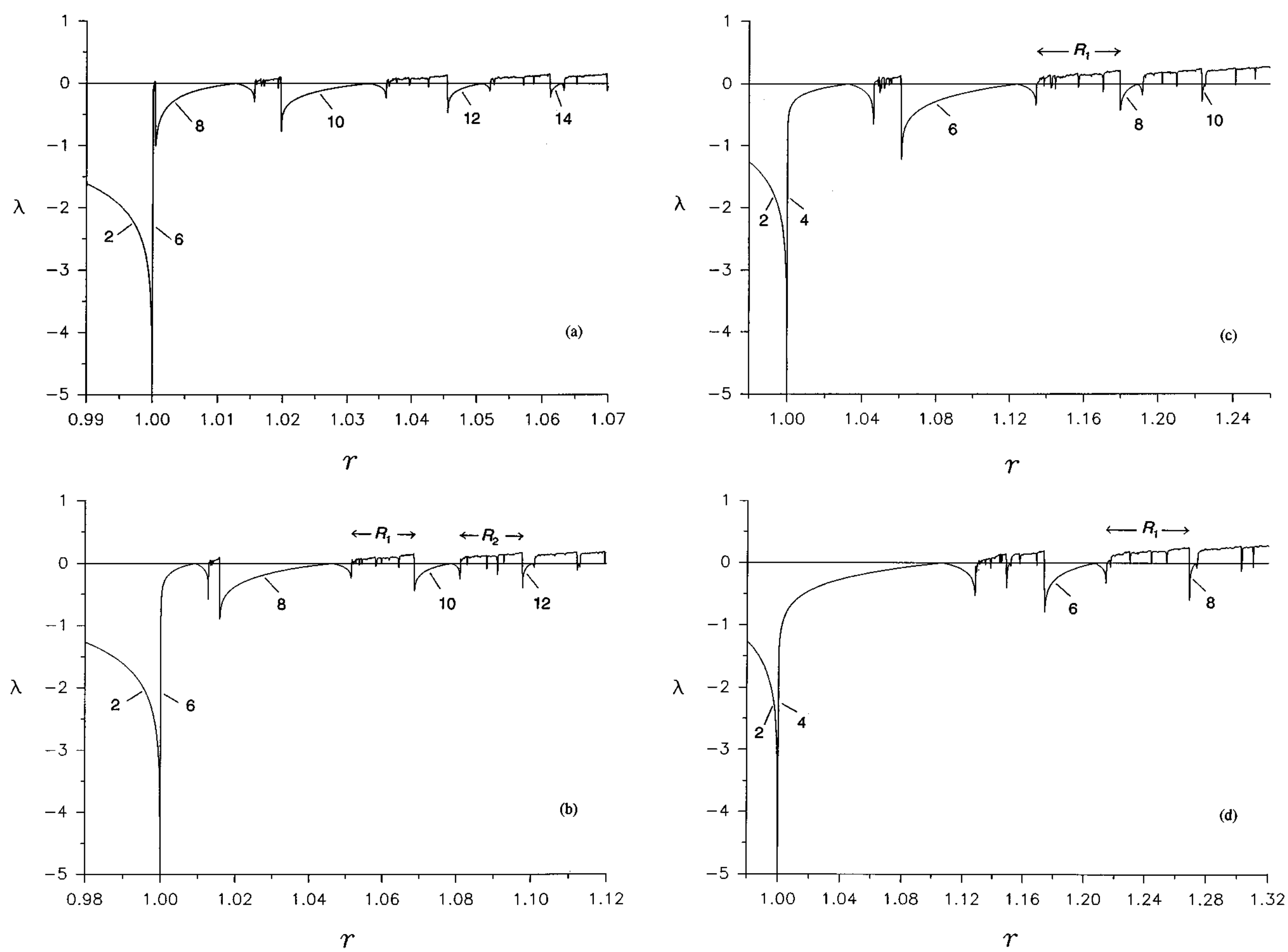

FIG. 2. Graph of Lyapunov exponent λ against parameter r for the power-logistic map with (a) $z=0.4$, (b) $z=0.5$, (c) $z=0.8$, and (d) $z=1.0$. The periods of the 2 cycle, the P cycle, and the earlier PA terms are labeled in order of increasing r.

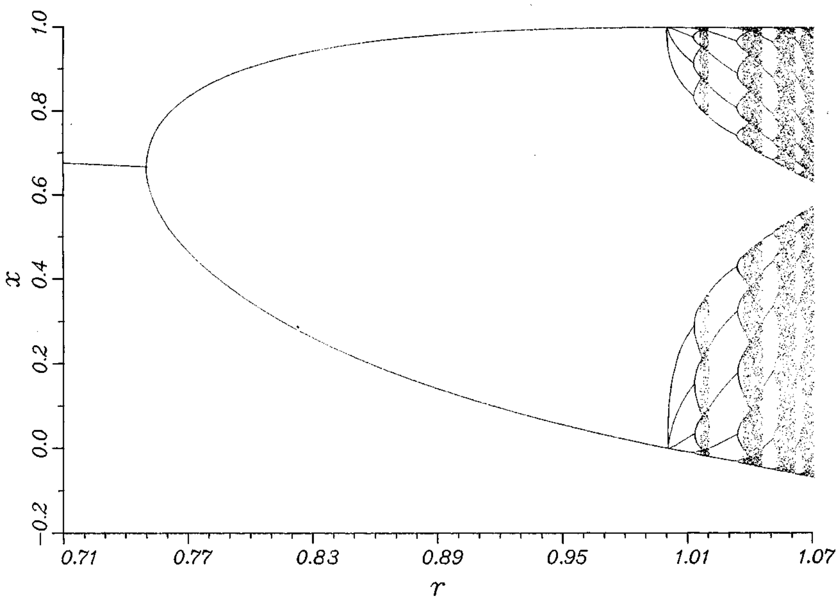

FIG. 3. Bifurcation diagram of the power-logistic map with $z=0.4$.

sion, we refer the reader to Figs. 2(b)–2(d), which are the λ-r graphs for the power-logistic maps with $z=0.5$, $z=0.8$, and $z=1.0$, respectively. Remarks similar to those made for Fig. 2(a) are also applicable to Figs. 2(b)–2(d). In these figures, the earlier PA terms, that is, the terms at the beginning of the series, are clearly marked; however, the later PA terms, namely, the other PA terms, are not visible in the above bifurcation diagram and λ-r graphs.

For a given value of z, this series, which lies in the chaotic region, is an arithmetic progression where the first term is an integer and the common difference is 2. Between two consecutive terms, there are small chaotic subregions and periodic windows containing other cycles including the period-doubled cycles of the term with smaller r. For convenience, we shall refer to this series as *pattern A* or PA for short, and denote it by PA(z), which represents $P_1, P_2, P_3, \ldots$, where $P_{i+1}=P_i+2$ and P_1 is an integer that is a function of z. The value of P_1 is also equal to $2+P$, where P is the period of the cycle in the periodic region, which begins to appear at the end of the 2 cycle; we shall refer to this periodic orbit as the P cycle, with P being clearly a function of z. For example, for $z=0.4,0.5,0.8,1.0$, the period of the P cycle is equal to 6,6,4,4, respectively, as can be seen in Figs. 2(a)–2(d) where the P cycle is the second labeled cycle in each figure.

B. Distribution of stable cycles and pattern *A* for $z=0.4$

In order to have a better understanding of the distribution of the stable cycles including the PA terms in parameter space, we shall consider the special case of the power-logistic map with $z=0.4$. In the periodic region of this particular map, as r increases, we can identify stable cycles with periods 1,2,6,12,24,28, ... Here 6 is the period of the P cycle while 12,24,48, . . . are the periods of its period-doubled cycles. On the scale of the bifurcation diagram given in Fig. 3 for this case, other than the 1 and 2 cycles, the other stable periodic orbits in the periodic region are not visible.

In the chaotic region, the first few PA terms, namely, the 8, 10, 12, and 14 cycles, are easily found as they occupy fairly large ranges of r. These are shown in Fig. 3 with each lying between two consecutive chaotic subregions. Note that in this figure, the first chaotic subregion that lies at the beginning of the chaotic region is not visible.

Each of the PA terms gives rise to period-doubled cycles just as the 6 cycle in the periodic region does. Hence, in the chaotic region there is an intermingling of the PA terms with their period-doubled cycles and chaotic subregions. For this case, the first PA terms with their associated period-doubled cycles are *8*,16,32, . . . chaos; *10*,20,40, . . . chaos, *12*,24,48, . . . chaos, where the PA terms are italicized.

There appear to be no other stable periodic orbits between these period-doubled cycles. However, other stable periodic orbits may appear in the chaotic subregion that lies between the end of each period-doubling sequence and the beginning of the next PA term.

C. Pattern *A* for some other values of *z*

Since both PA and its first term P_1 are functions of z, the appearance of the bifurcation diagram is z dependent, as shown in the bifurcation diagrams given in Figs. 1, 3, 4, and 5 for $z=0.8$, 0.4, 0.6, and 0.2, respectively. These diagrams show that the PA terms become less prominent as z increases.

The bifurcation diagram for $z=0.8$ given in Fig. 1 shows that $P=4$ in the periodic region, which implies that the first PA term has a period of 6. This term and the succeeding PA terms, 8,10, . . . can be identified in enlargements of Fig. 1.

For the case when $z=0.6$, the PA terms 6, 8, 10, and 12 as well as their first period-doubled cycles are shown in Figs. 4(a)–4(b). Here, the prominent periodic orbits are 1,2,*6*,12, . . . , *8*,16, . . . , *10*,20, . . . , *12*,24, . . . , where the italicized integers are the periods of the PA terms; the others are those of the period-doubled cycles of these PA terms. The P cycle, which has a period of 4 and which lies at the end of the 2 cycle, is not shown in Fig. 4(a) as its range is very narrow.

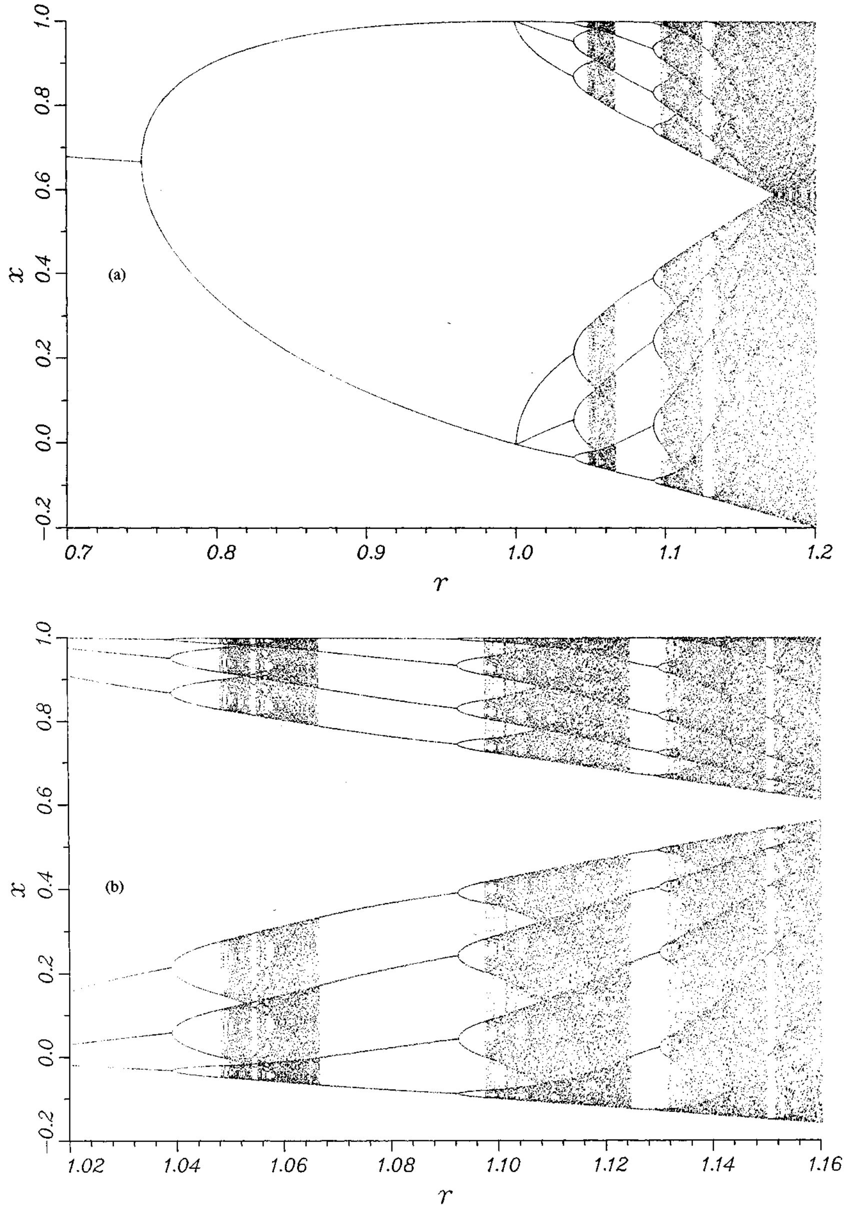

FIG. 4. (a) Bifurcation diagram of the power-logistic map with $z=0.6$. (b) Enlargement of part of the diagram in (a).

In the bifurcation diagram for $z=0.4$ shown in Fig. 3, each of the PA terms 8,10,12 and their respective period-doubled cycles occupies quite a large range. The prominent stable orbits are 1,2,*8*,16, . . . , *10*,20, . . . , *12*,24, . . . with the same convention used above. The P cycle, which has a period of 6 and which lies at the end of the 2 cycle, is not visible in Fig. 3 as it occupies a very tiny interval of r.

In the case of $z=0.2$, as each of the earlier PA term as well as its period-doubled cycle occupies a fairly large range, they are visible in Fig. 5. The prominent cycles are 1,2,*10*,20, . . . , *12*,24, . . . , *14*,28, . . . , *16*,32, . . . , *18*,36, . . . with the previous convention. As in the other cases, the P cycle, which has a period of 8 and which is located at the end of the 2 cycle, is not visible in this figure.

III. VARIATION OF PROPERTIES OF MAPS WITH z

In the λ-r graphs shown in Figs. 2(a)–2(d), the earlier PA terms are labeled and distinguished by their smoothly ascending structures. Between the end of a PA term, say P_i, and the beginning of the next PA term, P_{i+1}, is a chaotic subregion, to be denoted by R_i, in which there are period-doubled cycles and other periodic orbits intermingled with chaos. Some of these chaotic subregions are labeled in Figs. 2(b)–2(d).

A. Range of P

From the λ-r graphs for different values of z, say from 0.4 to 1, we observe that $\Delta r(P)$ increases with z, where $\Delta r(P)$ is defined to be the range of the P cycle that occurs just after the 2 cycle in the periodic region. For example, as z increases from 0.4 [Fig. 2(a)] to 0.5 [Fig. 2(b)], $\Delta r(P)$ increases where P is equal to 6. When z has increased to 0.516, a new 4 cycle appears between the 2 cycle and the 6 cycle. Hence for $z<0.516$, the 2 cycle is followed by a 6 cycle whereas for $z>0.516$, there is an additional 4 cycle, which is now the new P cycle.

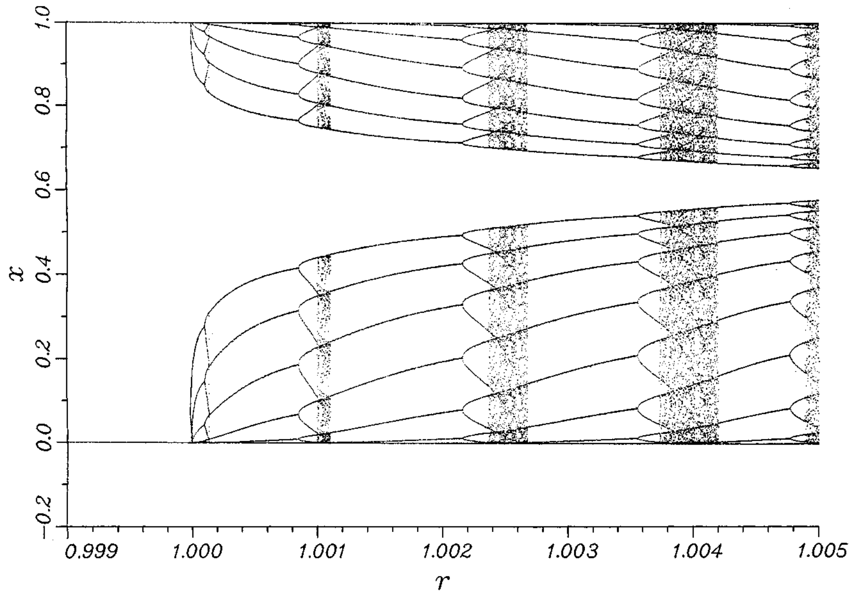

FIG. 5. Bifurcation diagram of the power-logistic map with $z=0.2$.

The range of the new 4 cycle also increases with z, while that of the 6 cycle may not continue to increase monotonically as can be seen from Figs. 2(c) and 2(d), which corresponds to $z=0.8$ and 1.0, respectively. Though the range of the 4 cycle when it first appears is smaller than that of the 6 cycle, it subsequently becomes larger.

B. Size of R_i

For small values of i, the separation R_i between the PA terms P_i and P_{i+1} also increases with z. This can be verified by comparing Fig. 2(b) for $z=0.5$ with Fig. 2(d) for $z=1.0$. In Fig. 2(b), $P=6$ and R_1, which separates the PA terms 8 and 10, is equal to 0.0169 while R_2, which separates the PA terms 10 and 12 is 0.0165. In Fig. 2(d), the PA terms that are visible are 6 and 8 with $P=4$ and $R_1=0.0528$, which is larger than R_1 of Fig. 2(b). However, not all the R_i's will increase monotonically with z when i is larger.

An increase of the first few R_i's together with that of $\Delta r(P)$ gives the impression that as z increases, these features in the middle portion of the λ-r diagram are displaced in the direction of increasing r. Thus as z increases, the PA terms $P_i, P_{i+1}, P_{i+2}, \ldots$ for a constant i begin to appear at larger values of r.

C. Existence of another pattern

We shall now consider the transition in the λ-r diagram as z increases from 0.8 to 1.0. As shown in Figs. 2(c) and 2(d), both the ranges of the PA terms P_1 and P_2 decrease but R_1 increases such that the beginnings of both the P_1 and P_2 windows occur later (i.e., at larger values of r), where $P_1=6$ and $P_2=8$. Moreover, the growing chaotic subregion just before P_1 develops more fine structures as z increases. This can be seen from Figs. 2(d) and 2(c) and their enlargements: when $z=1.0$, many more periodic orbits with smoothly ascending structures in the chaotic subregion between P and P_1 can be identified than when $z=0.8$. When $z=1.0$, it is observed that the periods of these new cycles form another arithmetic progression, namely, the series *8,16,24,32,40*, ... This new arithmetic progression has a common difference of 8 and it is located before PA.

D. Development of more features

As seen above, when i is small, both R_i and $\Delta r(P)$ increase with increasing z. Further, we observe that when i is small the range of P_i decreases as z increases from a sufficiently large value. Despite these opposing effects, we find (Sec. III B) that as z increases, the earlier portion of the chaotic region is displaced in the direction of increasing r such that the PA terms begin to appear later. One effect of this movement of the PA series in the direction of increasing r is that more ''room'' at the beginning of the chaotic region is made available to accommodate additional ''earlier'' PA terms when z becomes larger. This accounts for the creation of the 4 cycle between the 2 cycle and the 6 cycle when z increases from 0.4 to 0.516 (Sec. III A). As another example, when z increases from 0.4 to 0.5 [Figs. 2(a) and 2(b)], the PA series 10,12,14, . . . becomes 8,10,12,14, . . . with an additional 8 cycle. Likewise, when z has increased to 0.8, a new term, namely 6, appears with the series becoming 6,8,10,12,14, . . . instead [Fig. 2(c)].

Another consequence of the decrease of the range of P_i and the increases of both $\Delta r(P)$ and R_i for small i when z increases is that it is more difficult to identify the PA terms for they could occupy smaller ranges than the other non-PA terms especially when z becomes fairly large.

As seen in Sec. III C, more fine structures such as another arithmetical progression may develop as z increases. Thus, the λ-r graph as well as the bifurcation diagram will generally have more complicated features when z is large.

IV. EXISTENCE OF PA IN LOGISTIC MAP AND OTHER FEATURES

As seen in Secs. II C and III D, as z increases, the PA terms occupy smaller ranges, which means that PA becomes less prominent. From grounds of continuity we expect that

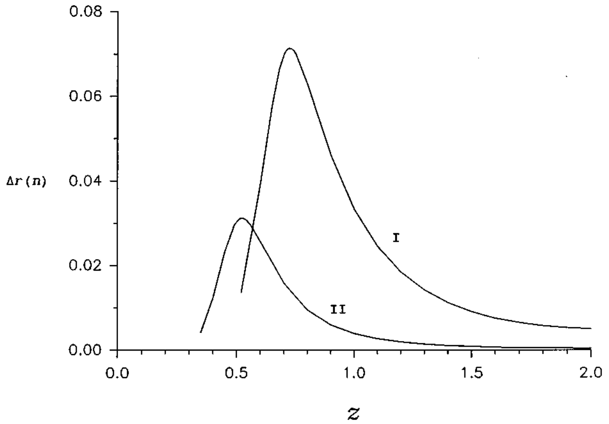

FIG. 6. Graph of $\Delta r(n)$ against z. Curves I and II represent $\Delta r(6)$ and $\Delta r(8)$, respectively.

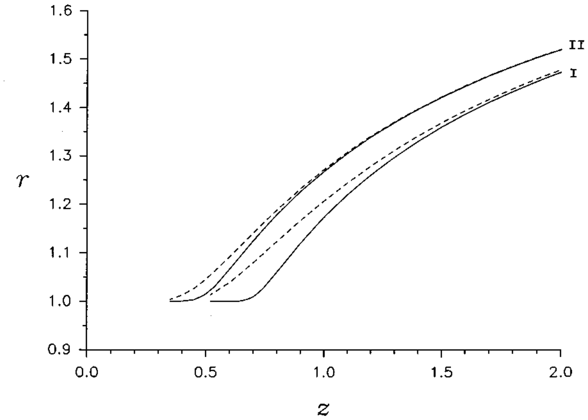

FIG. 7. Graphs of r(6,min), r(6,max), r(8,min), and r(8,max) vs z. The solid curves in I and II denote r(6,min) and r(8,min), respectively, while the dashed curves in I and II denote r(6,max) and r(8,max), respectively.

PA should exist for all values of $z \in (0,2]$. To test this idea, we shall examine the z dependence of $\Delta r(6)$ and $\Delta r(8)$, which are the ranges of the 6-cycle and 8-cycle PA terms, respectively.

A. Graphs of $\Delta r(6)$ and $\Delta r(8)$

$\Delta r(6)$ is a function of z as shown in Fig. 6. As z increases from about 0.5 to about 0.7, $\Delta r(6)$ increases rapidly to a maximum value of about 0.072. Subsequently, it decreases to a value of 0.005 as z increases to 2. The range of the 8-cycle PA term, $\Delta r(8)$, has a similar dependence on z: as z increases from about 0.3 to about 0.5, it increases to a maximum of about 0.032 from which it decreases to about 0.001 when $z=2$. It follows that for z in the above range, $\Delta r(6)$ and $\Delta r(8)$ are continuous functions of z and that the earlier PA terms can be identified even for the logistic map in which they occupy very tiny ranges.

We believe that the ranges of the later PA terms are also continuous functions of z though it is impossible to numerically verify this for large i since for large z, their ranges are of the order of 10^{-N_D}, where N_D is the maximum number of significant figures available in the computation. Thus as the earlier PA terms have finite ranges at $z=2$, it follows that at least the earlier terms of PA do exist in the logistic map, if note the whole series. In fact, the existence of PA in the logistic map has previously been reported [11].

Note that the steady decreases of $\Delta r(6)$ and $\Delta r(8)$ from their respective maximum with increasing z are consistent with the earlier observation in Sec. III D that when i is small the range of P_i decreases as z increases from a sufficiently large value.

B. Beginning and ending r values of 6 and 8 cycles

The graphs of r(6,min), r(6,max), r(8,min), and r(8,max) as functions of z for the power-logistic maps are shown in Fig. 7, where $r(n,\text{min})$ and $r(n,\text{max})$ denote, respectively, the minimum and maximum values of r when the n cycle exists. This figure shows that all these functions increase monotonically with z, a behavior that is consistent with the shift of the portion of the chaotic region with small values of r in the direction of increasing r as z increases.

As shown in Fig. 7, the smallest values of r(6,min) and r(8,min) are both 1, while the smallest value of z for which the 8 cycle exists is smaller than that for the 6 cycle. From this and the monotonic nature of these four functions, we can deduce that as z decreases, the earlier PA terms will be pushed in the direction of decreasing r until some of them cease to exist as they are ''pushed'' out of the chaotic region. The last deduction is consistent with the observation in Sec. III D that as z increases, there is more ''room'' at the beginning of the chaotic region to accommodate earlier PA terms.

It is convenient to view the above process as z decreases from 2 instead. As z decreases, $\Delta r(6)$ and $\Delta r(8)$ increase until z reaches about 0.7 and 0.5, respectively, while r(6,min), r(6,max), r(8,min), and r(8,max) all become smaller. This increase of the ranges and the earlier appearance of the beginning and ending of both the 6 and 8 cycles imply that these terms encroach towards the periodic region as z decreases from 2 to 0.7. In general, other PA terms are expected to behave similarly. Consequently, as z decreases, the first PA term will be larger since the earlier terms would have been ''pushed'' out of the chaotic region.

C. Relation between PA and period-adding series in cusp maps

From above, we see that PA, while is an arithmetic progression with a common difference of 2, exists in the chaotic region of the power-logistic map for all values of z in the range of interest, regardless of whether or not the slope of the map is continuous everywhere. This is to be contrasted with the existence of the ''period-adding'' phenomenon in cusp maps, which have discontinuous slopes at the cusps [12–14]. This phenomenon gives rise to a sequence of attracting periodic orbits in the chaotic region, which forms an arithmetic progression with a common difference of 1 instead.

Thus pattern A, which exists in the power-logistic map, is not identical to the period-adding series in the cusp maps, since the common difference is 2 in PA, and it does exist in maps which may or may not have continuous slopes everywhere.

V. SUMMARY AND DISCUSSION

The power-logistic map given by Eq. (1) for $z \in (0,2]$ has many interesting properties. Here we shall briefly summarize some of our main results and give a brief discussion on certain other aspects.

In the chaotic region of the power-logistic map for all values of z in the range of interest, there are some stable cycles, each distinguished by its smoothly ascending structure from a negative value of λ to a peak value of 0, with periods that can be regarded to form an arithmetic progression. This series is referred to as pattern A (PA), and it has a common difference of 2 with the first PA term, P_1 either remaining constant or increasing by 2 as z decreases. The value of P_1 is larger than that of the stable cycle occurring after the 2 cycle in the periodic region by 2.

When z is small, the PA terms are very prominent compared to other cycles and are therefore easily identifiable. As z increases, the range occupied by each PA term generally decreases when z is large enough; it is therefore more difficult to identify the PA terms for large z, especially in the case of the logistic map. In the latter case, the PA terms occupy very tiny ranges of r and are not as prominent as many other stable cycles in the chaotic region: hence the existence of PA can be easily overlooked here.

Is PA a finite or an infinite series? It is not possible to answer this question by using numerical methods since the later PA terms have very tiny ranges and hence cannot be isolated even if they do exist.

The bifurcation diagrams for the power-logistic map for different values of z are related to each other. In particular, as z increases, the range of the chaotic region decreases, with the same PA term commencing at a larger value of r and occupying a smaller range when z is sufficiently large. Despite the shrinking of the chaotic region as z increases, there is now more room at the beginning of the chaotic region to accommodate earlier PA terms.

It is convenient to describe the variations of certain properties of the bifurcation diagram as z decreases. The range of the periodic region decreases while that of the chaotic region increases. Within the latter region, the size of the chaotic subregion lying between any two consecutive PA terms decreases while that of the PA term increases when z is sufficiently large. We can say that the chaotic region shifts into the periodic region such that the beginning of any PA window occurs at a smaller value of r, and the earlier PA terms are squeezed out of existence. Hence as z decreases, the earlier PA terms as well as their period-doubled cycles may cease to exist: for example, when $z=0.6$, pattern A consists of the 6,8,10, . . . cycles, whereas when z has decreased to 0.5, the series does not include the 6 cycle. Further, the decrease in size of the chaotic subregion may result in a gradual disappearance of fine structures such as the series 8,16,24,32,40, . . . , which occurs in the chaotic region between the P cycle and the first PA term when $z=1$.

ACKNOWLEDGMENT

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